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Rigorous properties of the two-dimensional Ising model with periodically distributed frustration

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Abstract. The two-dimensional Ising model on the square lattice with a fraction of periodically distributed frustrated cells equal to 1, $\frac{2}{3}$, and $\frac{1}{2}$ is solved rigorously. It is shown that a phase transition, understood in the sense of a singularity in the specific heat, exists for the two latter model systems while it disappears for the completely frustrated lattice. The phase transition temperatures are lower than that of a ferromagnetic Ising model and increase with decreasing frustration. The mean-field approximation results suggest that long-range order is present in areas of nonfrustrated cells.

1. Introduction

The nature of phase transitions in spin glasses has recently been intensely investigated (for a review see e.g. Fisher 1977 and Kinzel and Fisher 1978). The experimental data show that random magnets differ qualitatively from pure systems. There is no singularity in the field-dependent properties of spin glasses, but at a certain characteristic temperature T_f a cusp in the static magnetic susceptibility appears. However, no well defined characteristic temperature has been observed in the thermal properties.

A phase transition in a spin glass is still a very puzzling matter. Monte-Carlo calculations of two-dimensional Ising spin glasses showed that the specific heat exhibits a very broad peak (Binder and Schröder 1976, Binder 1977, Kirkpatrick 1977a, b) whereas the susceptibility has a very sharp cusp. However, numerical techniques can never yield a definite proof that a phase transition either does or does not exist, nor does it yield any other rigorous results. On the other hand, the real-space renormalisation group technique gives only very inconclusive answers (Kinzel and Fisher 1978) while extrapolation of the high-temperature series expansions suggests that no phase transition occurs for dimensionality d below four (Fisch and Harris 1977). In this situation it is still a question of whether a phase transition appears in a two-dimensional spin glass, and, if so, how it should be understood.

In the theoretical description of spin glasses, a concept of relevant and irrelevant disorder has been introduced (Toulouse 1977, Vannimenus and Toulouse 1977, and Kirkpatrick 1977b). It is well known that there is a class of models, commonly known as Mattis models (Mattis 1976), for which randomness is trivial since it can be eliminated by a suitable gauge transformation. Therefore, Toulouse (1977) and Kirkpatrick (1977b) advanced the concept of frustration as a measure of relevant disorder. The idea

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of frustration is that not all the bond interactions around an elementary circuit in a lattice can be simultaneously satisfied. For a plane random Ising model this situation appears for a particular square while one of the bonds has a different sign from the other three. Consequently, the properties of a random magnet are dependent on the distribution of frustrated elementary cells and not on the ratio between positive and negative bonds (Fradkin *et al* 1978).

Frustrated cells are distributed randomly in a spin glass. Any theoretical approach to such a system has to deal with averaging over the probability distribution of frustrated cells. This is a very difficult problem in the theory of spin glasses (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975, Thouless *et al* 1977) and, therefore, the concept of frustration can only be rigorously studied on some simple model systems. Such a study has recently been performed by Derrida *et al* (1978) for the random-bond and random-field one-dimensional Ising chains, and the small random-bond systems of spins in two dimensions. Their results indicate existence of a singularity in dS/dx in two dimensions, where x is a concentration of negative bonds, which is interpreted as a ferromagnet–spin-glass phase transition with increasing concentration of negative bonds.

As emphasised by Toulouse (1977) it is necessary to study the frustration effect in its own right, at first in simple cases unobscured by all the intricacies of real spin glasses. In this paper we report the exact results obtained for two-dimensional Ising model systems with periodically distributed lines of frustrated cells. These models contain 100%, 67%, and 50% of frustrated cells, respectively. The partition function is found rigorously in a frame of the dimer method, as derived in § 2. The internal energy, the specific heat and the entropy are next calculated and displayed in § 3. With the mean-field approximation, introduced in § 4, we comment on the problem of the existence of long-range order in the systems with frustration. The final remarks and discussion are given in § 5.

2. Models with periodically distributed frustration and the dimer method

The Hamiltonian of a two-dimensional Ising model reads

$$H = -\sum_{ij}{}' J_{ij} S_i^z S_j^z, \tag{1}$$

where primed summation excludes the terms with $i > j$, $S_i^z = \pm 1$, and J_{ij} is the exchange interaction constant between the spins located at sites i and j. Let us assume that it takes only two values: J and $-J$. If all the interaction constants are equal to J, one has a perfect ferromagnet. Replacement of any positive bond by a negative bond produces two adjacent frustrated cells. On the other hand, if the sign of J_{ij} is reversed on all bonds around one site, this is not a serious disorder because, by flipping the spin on that site, one obtains a spin configuration which has the same energy as in the perfect ferromagnet. This property is a consequence of the gauge invariance of the Hamiltonian (1) with respect to such a transformation.

Here we will consider only the models in which reversing the signs of some J_{ij} in the Ising ferromagnet leads to a relevant disorder. A model in which all the cells are frustrated can be obtained if the bonds on every second line in one (say vertical) direction are considered negative whereas all the other bonds are positive. Such a model was previously discussed by Villain (1977) and will henceforth be called the

Villain model. Less frustrated cells are obtained, if the bonds on every third or fourth vertical line are negative, which give a ratio of $\frac{2}{3}$ and $\frac{1}{2}$ of frustrated cells. These two systems will be called B and C models, respectively. We present the unit cells of all three models in figure 1.

Thermodynamic properties of the models introduced above can be obtained with the help of the so called dimer method developed for the perfect Ising model by Kasteleyn (1963) and Montroll *et al* (1963). Calculation of the partition function is in this method reduced to the combinatorial problem of dimers occupying neighbouring sites of a lattice graph. Then the partition function of the Ising model can be evaluated as a Pfaffian $P(A)$, where A is a $4N \times 4N$ matrix constructed for a lattice of the size $N \times N$. The Pfaffian itself is the square root of a skew-symmetric determinant. For the details of this method we refer the reader to the original papers (Kasteleyn 1963, Montroll *et al* 1963).

For the models with periodically distributed frustration the matrix A has a cyclic structure with N identical non-zero matrices of dimensionality $4m \times 4m$ standing on its diagonal, where N is the number of elementary cells (see figure 1) and m is the number of lattice sites per elementary cell. Therefore, the expression for the free energy per spin reads

$$-\frac{F}{k_B T} = \lim_{N\to\infty} \frac{1}{mN} \ln Z$$

$$= \ln(2\cosh^2 K) + \frac{1}{8m\pi^2} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \ln|\det \lambda(\phi_1, \phi_2)|, \qquad (2)$$

where $\lambda(\phi_1, \phi_2)$ is a $4m \times 4m$ matrix defined below.

The matrix $\lambda(\phi_1, \phi_2)$ consists of 4×4 matrices by which a solution of a perfect Ising ferromagnet can also be expressed. Let us define the first five matrices $a(i, j)$:

$$a(0,0) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix},$$

$$a(1,0) = -a^{\mathrm{T}}(-1,0) = \begin{pmatrix} 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3)$$

$$a(0,1) = -a^{\mathrm{T}}(0,-1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $z = \tanh J/k_B T$. The elements of $\lambda(\phi_1, \phi_2)$ will be defined with the help of matrices Λ_i,

$$\Lambda_1 = a(0,0) - a(0,1)\exp(i\phi_2) - a(0,-1)\exp(-i\phi_2),$$

$$\Lambda_2 = a(0,0) + a(0,1)\exp(i\phi_2) + a(0,-1)\exp(-i\phi_2),$$

$$\Lambda_3 = a(1,0),$$

$$\Lambda_4 = a(-1,0). \qquad (4)$$

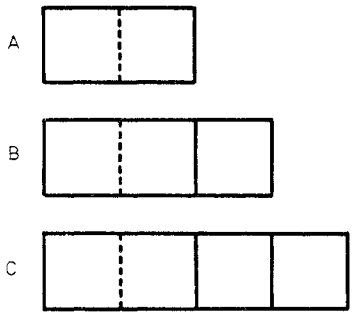

Figure 1. Unit cells for the Villain model (A), B model (B), and C model (C). Full bonds correspond to $J_{ij}=J$ and broken bonds correspond to $J_{ij}=-J$.

Then one has for the Villain model

$$\lambda(\phi_1,\phi_2)=\begin{pmatrix}\Lambda_1 & \Lambda_3+\Lambda_4\,e^{-i\phi_1}\\ \Lambda_3\,e^{i\phi_1}+\Lambda_4 & \Lambda_2\end{pmatrix}, \tag{5}$$

for the B model

$$\lambda(\phi_1,\phi_2)=\begin{pmatrix}\Lambda_1 & \Lambda_3 & \Lambda_4\,e^{-i\phi_1}\\ \Lambda_4 & \Lambda_2 & \Lambda_3\\ \Lambda_3\,e^{i\phi_1} & \Lambda_4 & \Lambda_2\end{pmatrix}, \tag{6}$$

and for the C model

$$\lambda(\phi_1,\phi_2)=\begin{pmatrix}\Lambda_1 & \Lambda_3 & 0 & \Lambda_4\,e^{-i\phi_1}\\ \Lambda_4 & \Lambda_2 & \Lambda_3 & 0\\ 0 & \Lambda_4 & \Lambda_2 & \Lambda_3\\ \Lambda_3\,e^{i\phi_1} & 0 & \Lambda_4 & \Lambda_2\end{pmatrix}. \tag{7}$$

The determinants of the matrices $\lambda(\phi_1,\phi_2)$ are polynomials in z and trigonometric functions of the integration variables ϕ_i. They are given below for
1) the Villain model:

$$\det\lambda(\phi_1,\phi_2)=(z^2-1)^2[(z^2+1)^2-2z^2(\cos 2\phi_2+\cos\phi_1)], \tag{8}$$

2) the B model:

$$\begin{aligned}\det\lambda(\phi_1,\phi_2)&=-2z^3(1-z^2)^3\cos\phi_1+16z^5(1-z^2)\sin^2\phi_2\cos\phi_2\\&\quad+z^6(z^2+2z\cos\phi_2+1)[(1+z)^2-4z^2\cos^2\phi_2-4\sin^2\phi_2]\\&\quad+(z^2-2z\cos\phi_2+1)[(1+z^2)^2-4z^2\cos^2\phi_2-4z^4\sin^2\phi_2],\end{aligned} \tag{9}$$

and
3) the C model:

$$\begin{aligned}\det\lambda(\phi_1,\phi_2)&=-2z^4(1-z^2)^3\cos\phi_1+4z^4(1-\cos\phi_2^2)\\&\quad\times[z^2(z^2+2z\cos\phi_2+1)+z^2-2z\cos\phi_2+1](z^2+2z\cos\phi_2+1)\end{aligned}$$

$$-[4z^2\sin^2\phi_2+(z^2-2z\cos\phi_2+1)^2][(4\cos^2\phi_2-3)z^2-1]$$
$$-z^8[(z^2+2z\cos\phi_2+1)^2+4\sin^2\phi_2](z^2-4\cos^2\phi_2+3)$$
$$-4z^2\sin^2\phi_2(z^2-2z\cos\phi_2+1)(z^4+2z^3\cos\phi_2+2z^2-2z\cos\phi_2+1). \tag{10}$$

We notice that equation (8) leads to a different expression for the free energy from that derived by Villain (1977).

3. Thermodynamic quantities

We use the dimensionless units for the free energy $f=-F/k_BT$ and the exchange interaction constant $K=J/k_BT$. In these units one obtains:
1) the internal energy of the system

$$E/J=-\partial f/\partial K, \tag{11}$$

2) the specific heat

$$C/k_B=K^2(\partial^2 f/\partial K^2), \tag{12}$$

and
3) the entropy

$$S/k_B=f-K(\partial f/\partial K). \tag{13}$$

As the explicit expressions for all these quantities (11)–(13) are rather complicated and they may be straightforwardly derived from equations (8)–(10), we present here only the numerical results for all cases and compare them with the Ising model.

In figure 2 the internal energy is shown. At $T=0$ the internal energy for the models containing frustrated cells is greater than that for the Ising model as some of the bonds cannot be satisfied. The ground state energy E_G can be easily related to the fraction of

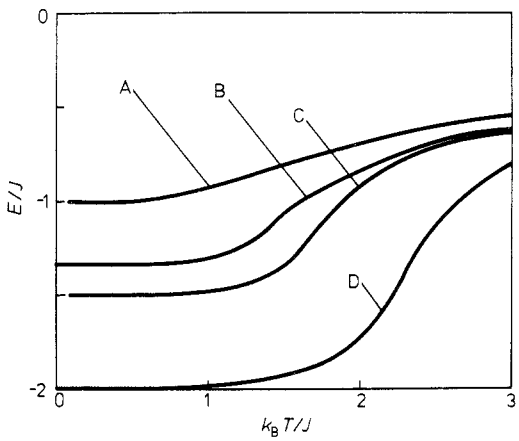

Figure 2. Internal energy for the Villain model (A), B model (B), C model (C), and the Ising model (D).

frustrated cells c by the equation

$$E_G = -2J(1-c/2). \tag{14}$$

In the most frustrated case the internal energy slowly grows with temperature and recalls the temperature dependence of energy of the antiferromagnetic triangular Ising model which is also a model system with all the cells being frustrated (Wannier 1950). For the remaining models a region of faster growth of the internal energy can be distinguished. This is revealed in the specific heat which is presented in figure 3. For the completely frustrated model the specific heat has a rather broad maximum at the temperature $k_B T/J = 0{\cdot}7$. In this case there is then no phase transition and any long-range order is hardly expected (see also Villain 1977). On the other hand, if only one third of the lattice cells is not frustrated, a singularity in the specific heat already appears. This singularity clearly suggests that a system exhibits a phase transition with some kind of long-range ordering below the transition temperature. If one half of the lattice cells is frustrated, the singularity comes out at a higher temperature. For both the B and C models, the specific heat singularity with the critical exponent $\alpha = 0$ is of the same type as in the Ising model, while peaks in the specific heat are more narrow for the systems with frustration.

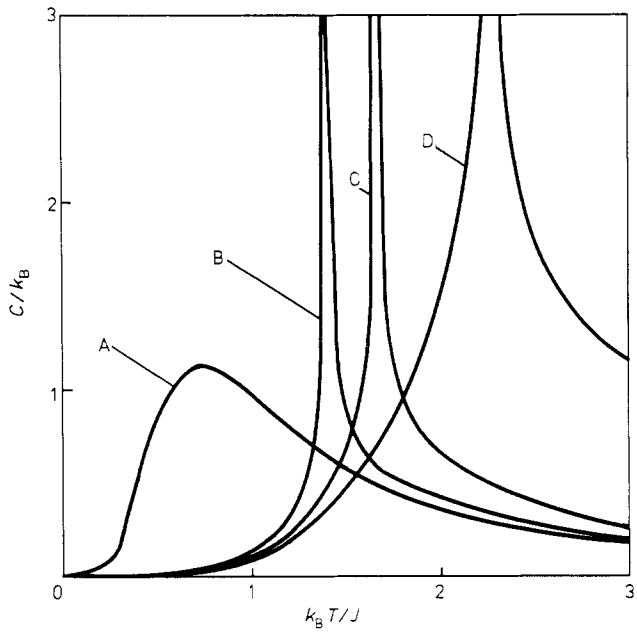

Figure 3. Specific heat for the Villain model (A), B model (B), C model (C), and the Ising model (D).

In all three cases the ground state of the system is strongly degenerated. This degeneracy can be expressed by the number of minimum length pairings of the frustrated cells (Kirkpatrick 1977b). For the Villain model we reproduce the exact result of Kasteleyn (1961) $S(T=0) = 0{\cdot}291k_B$. For the other models the entropy at $T=0$ is $0{\cdot}178k_B$ and $0{\cdot}082k_B$, for the B and C models, respectively. The temperature dependence of the entropy is shown in figure 4.

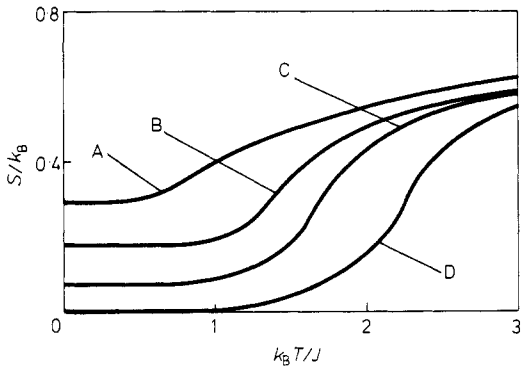

Figure 4. Entropy for the Villain model (A), B model (B), C model (C), and the Ising model (D).

4. Molecular field approximation

In order to understand the ground state of the considered models better, we have also studied the molecular field approximation equations for the averages $\langle S_i^z \rangle$.

A few formulations of the mean-field theory in spin glasses exist. The simplest Weiss approximation fails in reproducing the spin-glass solution (Southern 1976). Therefore, more elaborate approximations have been developed by Kaneyoshi (1975) and Southern (1976). Kinzel and Fisher (1977) divided spins in a spin glass into two classes of spins according to their orientation and later performed averaging over the probability distribution for molecular fields. This method was recently shown to be equivalent to the replica method (Chalupa 1978) which was commonly used to find the free energy of a random system.

In all the above formulations there is one additional approximation introduced as compared with the standard mean-field theory. In order to make the problem tractable, a set of selfconsistent equations for all the averages $\langle S_i^z \rangle$ in the crystal is replaced by one equation with an additional averaging over the molecular field with an appropriate probability distribution. This assumption allowed Kinzel and Fisher (1977) to derive a relationship between the 'freezing' temperature T_f and the concentration of impurity bonds which may be linear or square-quadratic for simple distributions of the exchange bonds. In fact, the former relationship is possible only while all the bonds are satisfied, i.e. there is no frustration effect. The 'freezing' temperature can in both cases be expressed by the first moment of the distribution of effective couplings (Kinzel and Fisher 1977).

Let us concentrate for a moment on the Villain model and treat all the neighbouring spins of a particular site as equivalent and put $\langle S_i^z \rangle = \langle S^z \rangle$. A molecular field acting on this site may be equal to $4J\langle S^z \rangle$ or $2J\langle S^z \rangle$ or zero. Here we only consider the situation when all spins are oriented according to the molecular field acting on them as this is the only case which leads to the correct ground state energy with the least possible number of exchange interactions unsatisfied (see figure 2 for $T = 0$). One of these arrangements is a configuration in which a field with absolute value equal to $2J\langle S^z \rangle$ acts on each spin. It means that exactly one bond coming out of each site is unsatisfied. However, such a configuration of spins cannot in practice stabilise because the entropy of such a state is very small, contrary to the exact results given in figure 4. The only freedom in such a

state is a distribution of unsatisfied bonds over a lattice with the constraint that any two unsatisfied bonds cannot join together.

Consider the opposite situation when half of the spins experience a molecular field equal to $4J\langle S^z\rangle$ and the other half are oriented stochastically as the field acting on them is equal to zero. For such a spin arrangement the degeneracy is equal to the product of the number of possible distributions of $Nm/4$ spins which have two of their neighbours oriented independently of the bonds, by the number of all possible configurations of these spins, which is $2^{Nm/4}$. In this case the unsatisfied bonds form chains in the lattice. These chains may either be closed or have free ends. One obtains the simplest realisation of such a structure when all spins point upwards. This is one of many configurations included in the degenerate ground state. The negative bonds in this state are not satisfied. Let us call spins lying on these bonds A spins and the other spins B spins. The free energy of the Villain model then reads

$$F/k_BT = -\ln 2 - \tfrac{1}{2}\{\ln\cosh 2K(\langle S_B^z\rangle - \langle S_A^z\rangle) + \ln\cosh 2K(\langle S_B^z\rangle + \langle S_A^z\rangle)\} + K\langle S_A^z\rangle\langle S_B^z\rangle + \tfrac{1}{2}K(\langle S_B^z\rangle^2 - \langle S_A^z\rangle^2), \tag{15}$$

and the molecular field equations are

$$\langle S_A^z\rangle = \tanh 2K(\langle S_B^z\rangle - \langle S_A^z\rangle),$$
$$\langle S_B^z\rangle = \tanh 2K(\langle S_B^z\rangle + \langle S_A^z\rangle). \tag{16}$$

These equations give the transition temperature $T_c = 2\sqrt{2}J/k_B$.

Similarly the models with every third and fourth vertical line negative may be considered. In the former case we call the spins on the lines of negative bonds A spins and all the other spins B spins. The free energy is

$$F/k_BT = -\ln 2 - \tfrac{1}{3}[\ln\cosh 2K(\langle S_B^z\rangle - \langle S_A^z\rangle) + 2\ln\cosh K(3\langle S_B^z\rangle + \langle S_A^z\rangle)] + \tfrac{1}{3}K(3\langle S_B^z\rangle^2 - \langle S_A^z\rangle^2) + \tfrac{2}{3}K\langle S_A^z\rangle\langle S_B^z\rangle, \tag{17}$$

and the critical temperature is found from the system of equations

$$\langle S_A^z\rangle = \tanh 2K(\langle S_B^z\rangle - \langle S_A^z\rangle),$$
$$\langle S_B^z\rangle = \tanh K(3\langle S_B^z\rangle + \langle S_A^z\rangle). \tag{18}$$

In the latter model we introduce three different types of spins. Spins lying on the lines of negative bonds we call A spins, their neighbours B spins, and the spins which have no neighbour on the line of negative bonds, C spins. The free energy for this system is

$$F/k_BT = -\ln 2 - \tfrac{1}{4}[\ln\cosh 2K(\langle S_B^z\rangle - \langle S_A^z\rangle) + 2\ln\cosh K(\langle S_A^z\rangle + 2\langle S_B^z\rangle + \langle S_C^z\rangle) + \ln\cosh 2K(\langle S_B^z\rangle + \langle S_C^z\rangle)] + \tfrac{1}{2}K\langle S_A^z\rangle\langle S_B^z\rangle + \tfrac{1}{2}K\langle S_B^z\rangle\langle S_C^z\rangle + \tfrac{1}{4}K(2\langle S_B^z\rangle^2 + \langle S_C^z\rangle^2 - \langle S_A^z\rangle^2), \tag{19}$$

and it gives the molecular field equations

$$\langle S_A^z\rangle = \tanh 2K(\langle S_B^z\rangle - \langle S_A^z\rangle),$$
$$\langle S_B^z\rangle = \tanh K(\langle S_A^z\rangle + 2\langle S_B^z\rangle + \langle S_C^z\rangle),$$
$$\langle S_C^z\rangle = \tanh 2K(\langle S_B^z\rangle + \langle S_C^z\rangle). \tag{20}$$

In table 1 we compare the mean field values of the transition temperature with their exact values. As usual in the mean-field approximation, the transition temperatures are

Table 1. Comparison between the values of the transition temperature obtained exactly and with the help of the molecular field approximation in units $4J/k_B = 1$.

	Percentage of frustrated cells	Transition temperature	
		Mean field	Exact
Villain model	100	0·7071	0·187†
B model	66·7	0·8431	0·342
C model	50	0·9051	0·410
Ising model	0	1·0000	0·5673

† For the Villain model this value corresponds to the maximum in the specific heat.

overestimated, but they decrease with increasing frustration, in a similar way to the exact values.

The order parameters are shown in figures 5–7. A solution with all the averages $\langle S^z \rangle$ different from zero appears below the phase transition temperature and has a lower energy than that of a paramagnetic solution. For the spins lying on the unfulfilled bonds these averages slowly increase with decreasing temperature below the point of a phase transition, and only near $T = 0$ do they grow to reach the zero-temperature limit $\langle S_A^z \rangle = 1$. On the other hand, temperature dependence of the other averages ($\langle S_B^z \rangle$ and $\langle S_C^z \rangle$) is similar to that of the perfect Ising ferromagnet in the molecular field approximation. This suggests that regions of spins which always keep the same relative orientation in all possible ground states, called by Derrida *et al* (1978) 'packets', may appear in between the nonfrustrated bonds. In fact, the molecular field approximation gives only the solution with all spins oriented upwards at $T = 0$.

Of course, there is no reason why the unfulfilled bonds should lie on the lines of negative interactions. For the completely frustrated model they may be distributed over the whole lattice. For the models with less frustration the bonds around nonfrustrated plaquettes must be fulfilled in the ground state as the ground-state energy satisfies condition (14). This limits the unfulfilled bonds only to regions of frustrated cells.

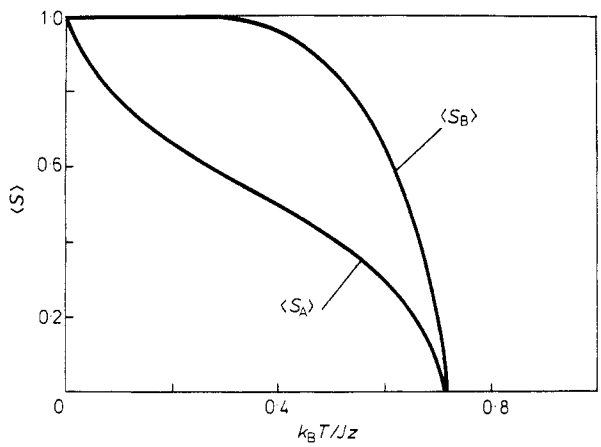

Figure 5. Selfconsistent solution of the molecular field equations for the Villain model. $z = 4$.

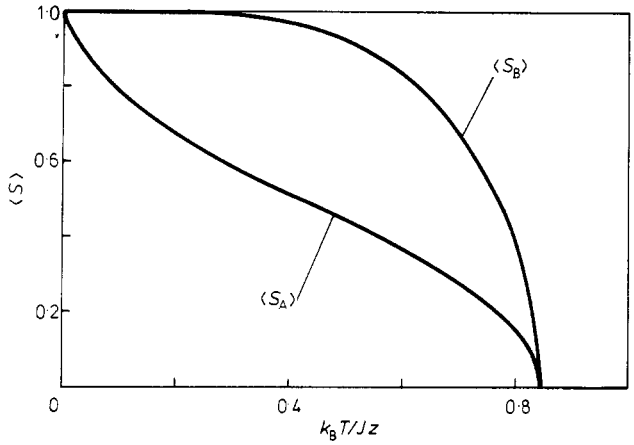

Figure 6. Selfconsistent solution of the molecular field equations for the B model. $z = 4$.

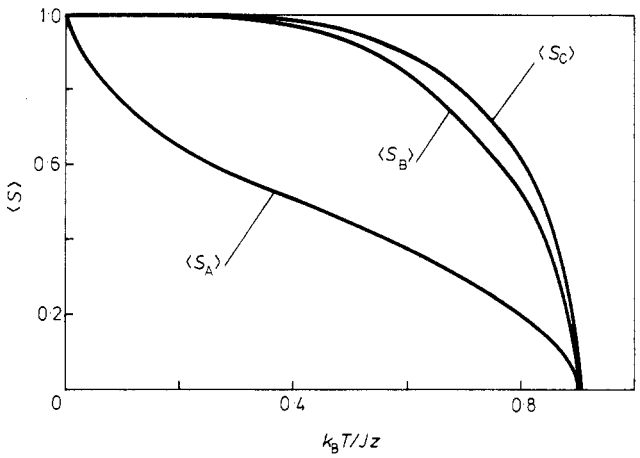

Figure 7. Selfconsistent solution of the molecular field equations for the C model. $z = 4$.

5. Final remarks and discussion

We have shown that the presence of frustration changes drastically properties of the two-dimensional Ising model. For the two systems containing strips of nonfrustrated cells, the B and C models, a singularity in the specific heat appears, while it is not present for the Villain model. This singularity exists at a lower temperature than for the ferromagnetic Ising model and has a smaller amplitude. The behaviour of the specific heat is here similar in both these aspects to that obtained within the same method for the Ising model containing regularly distributed impurities (Au-Yang *et al* 1976, Au-Yang 1976).

The models considered should show no net magnetic moment due to different orientations in different 'packets' of spins. On the other hand, a certain configuration of spins from many with the same energy stabilises at low temperatures. The symmetry of the system is broken by that particular spin arrangement and transitions to the other states need rather higher energy to generate intermediate states (for instance an energy

necessary to reverse all spins within one ordered area). At this point there is no conceptual difference between the models with periodically distributed frustration and a spin glass. Therefore, this suggests that a spin glass can exist in the two-dimensional Ising model in the same sense. A similar conclusion was recently reached by Stauffer and Binder (1978) who have shown by a Monte Carlo calculation that the ground state of the Edwards–Anderson spin-glass is highly degenerated and that the symmetry of the order parameter is quasicontinuous.

However, in a spin glass, frustrated cells are distributed randomly and its properties should be related to the random distribution of frustration (Fradkin *et al* 1978). Then a broad maximum observed in the specific heat may be due to averaging over different local distributions of frustrated cells. The Monte Carlo calculations of Vannimenus and Toulouse (1977) and recent series expansion in powers of negative bonds concentration x by Grinstein *et al* (1979) suggest that ferromagnetism is destroyed for $x \simeq 0{\cdot}09$. At higher concentrations a new phase (spin glass) forms with no long-range order. A similar transition has been observed for small spin systems (Derrida *et al* 1978). In the Villain model also only short-range correlations should be present. These correlations are due to satisfied bonds as in the ground state three bonds around each cell are fulfilled. On the contrary, for the B and C models there are long-range correlations within one 'packet' while no between spins belonging to two different 'packets' are expected. This possibility of a long-range order is also indicated by the molecular field approximation, although it gives a rather poor estimate for the transition temperature. To clarify this problem further we will present a future report of our results for the correlation functions.

We hope that rigorous solutions of the two-dimensional Ising model with periodically distributed frustration will contribute to a better understanding of spin glasses. We should mention that if spin ordering in spin glasses exists only on short distances, as in the Villain model, any renormalisation group transformation should be very carefully constructed in order not to average over important correlations.

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